ON THE WEDGING OF BRITTLE BODIES. SELF-OSCILLATIONS DURING WEDGING

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The static formulation of the problem of the wedging of brittle or quasi-brittle bodies by a rigid semi-infinite wedge, which corresponds to the case when the wedge velocity in the body is small compared with the velocity of propagation of transverse oscillations, has been studied in [1,2]. In dynamic form, which applies when the wedge velocity is comparable with this propagation velocity, the problem has been investigated in [3]. In both cases one of the main assumptions made is that the wedging is steady, i.e. as the wedge moves through the body with a constant velocity V, the leading edge of the crack formed in front of the wedge also moves uniformly at the same velocity V (Fig. 1).

Experimental investigations have shown, however, that the growth of a crack at low velocity does not take place uniformly (see [4]); the velocity of the tip of the crack performs regular oscillations about some mean value. At the same time the surface of the crack assumes a wave form. On the other hand, it happens that at a sufficiently high velocity of propagation the leading edge of the crack moves at a constant velocity and the surface of the crack becomes mirror-smooth.

In this connection the experiments in wedging carried out by Gilman, Knudsen and Walsh [5], and especially those by Kosevich [6], are very informative. These investigators found that at a sufficiently low wedging velocity the assumption that the velocity of the leading edge of the crack is constant is invalid. Gilman, Knudsen and Walsh discovered that at a sufficiently low wedging velocity in a crystal of lithium fluoride the leading edge of the crack moves in a series of jerks.

Kosevich discovered that when crystals of bismuth are split slowly along a plane of cleavage, the crack assumes an undulating surface, the undulations being symmetrical about the plane of the crack. The direction of the lines of the ridges has no definite crystallographic orientation but is always perpendicular to the direction of propagation of the crack. In addition, the velocity of the tip of the crack performs periodic oscillations about a mean value equal to the velocity of the wedge. This phenomenon is no longer observed when the velocity of the wedge increases to some critical value which is considerably less than the velocity of propagation of transverse oscillations.

Measurement of the distribution of the dislocation density on the surface of the split revealed the periodicity of this distribution, and it was found that the maxima of the dislocation density corresponded to the peaks of the ridges on the surface of the split. In [5,6] it has been stated that the non-uniform development of a crack can be explained by alternate brittle fracture and plastic deformation during the process of crack propagation.

In the present paper we shall expound the theory of a self-oscillatory process which occurs during wedging. The explanation of the phenomenon is based on the assumption that the cohesion modulus - the main characteristic of the forces acting at the tip of the crack [7] - depends on the instantaneous velocity of the tip of the crack, and initially decreases with increase in this velocity. The reduction in the cohesion modulus with increase in the velocity of crack propagation is typical of the quasi-brittle fracture of a great many materials and is associated with the decrease in inelastic strain in the layer adjacent to the surface of the crack.

In accordance with experimental data we can assume that the range of wedging velocities over which self-oscillation takes place is limited - at any rate for bodies of sensible dimensions - to velocities considerably less than the velocity of propagation of transverse oscillations.

1. Statement of the problem and the basic equation. 1. Suppose that a very large ideally elastic body is split under conditions of plane deformation (Fig. 1) by a rigid symmetrical wedge moving at a constant velocity V much less than the velocity of propagation of transverse oscillations c. Friction forces acting on the sides of the wedge will be ignored. Assume that the thickness of the wedge increases monotonically to a maximum value of $2h$. The axes of a moving system of Cartesian coordinates with origin at the tip of the crack will be denoted by ξ, η; those of a fixed system of coordinates will be denoted by x, y. The x and ξ axes lie in the direction opposite to the motion of the wedge. The instantaneous length of the free crack in front of the wedge will be denoted by l. Then, obviously,

$$\xi = x + \int v\,dt, \quad \eta = y \qquad (1.1)$$

where $v(t)$ is the instantaneous velocity of the tip of the crack and t is the time.

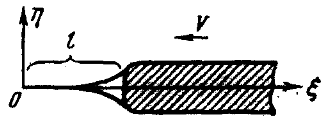

Fig. 1.

The proposed explanation of the self-oscillatory phenomena observed when a solid body is split, is based upon the assumption that fracture is quasi-brittle. This means that during fracture a thin layer of inelastic deformation is formed next to the surface of the crack.

As the crack propagates a given point on the surface of the crack is subjected to intense forces as the tip of the crack passes. The degree of inelastic deformation at this point increases with increase in intensity and duration of these forces. With increase in velocity the duration of these forces decreases. In the case of low velocities, when viscous effects are insignificant, the load intensity remains approximately constant. At high velocities, when viscous effects become considerable, the load increases. Therefore, as the velocity v of the tip of the crack increases the extent of inelastic deformation falls at first, and then, evidently, increases.

As a measure of the inelastic deformation it is customary to take the basic integral characteristic of the forces acting in the end region of the crack, i.e. the cohesion modulus K (see, for example [7]). In the case of quasi-brittle fracture the surface of the crack is taken as the boundary between the elastic and inelastic regions. The forces exerted on the elastic body by the discarded region of inelastic deformation are external forces with respect to the elastic body. Therefore, the cohesion modulus K increases with increase in inelastic deformation. Bearing in mind what we have said concerning the variation in inelastic deformation with increase in velocity, we assume* that the cohesion modulus at first decreases (until $v = v_*$) and then increases (Fig. 2). The critical velocity can be both much less than and comparable with the velocity of sound.

An analogous relation can be obtained between the velocity v and the

* The assumption that the cohesion modulus (or the density of surface energy) depends on the velocity of the tip of the crack has been expressed in the context of other problems in a paper by Stroh [8] and, independently, by V.A. Maksimov.

density of the surface energy T, which is related to the cohesion modulus K by the expression (see, for example, [7])

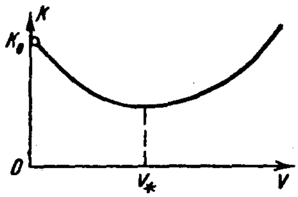

Fig. 2.

$$T(v) = \frac{(1-\nu^2)K^2(v)}{\pi E} \tag{1.2}$$

where E is Young's modulus and ν is Poisson's ratio for the material penetrated by the wedge. There is a characteristic similarity in the relations between cohesion modulus and time and those between coefficient of Coulomb friction and time; this similarity is one illustration of the close analogy between the processes of friction and crack formation.

The velocity v of the tip of the crack can be expressed in the form

$$v = V + \frac{dl}{dt} \tag{1.3}$$

We recall also that in the case of steady splitting of a body by a rigid smooth slow-moving wedge of constant thickness the length l of the free crack is defined by the static expression [2]

$$l = \frac{E^2h^2}{4(1-\nu^2)^2K^2} \tag{1.4}$$

2. To derive the basic equation defining the relation between the length of the free crack and time we start from the law of conservation of energy, which for the present problem can be written in the form

$$\frac{d\mathscr{E}}{dt} + \frac{d\Pi}{dt} = FV - 2T(v)\,v \tag{1.5}$$

where $\mathscr{E}$ is the kinetic energy of the body, Π its potential energy and F the absolute magnitude of the splitting force exerted by the wedge on the body and directed, obviously, along the axis of the wedge in the direction of motion. The quantity FV represents the work done by the external forces acting on the body over unit time; the quantity $2T(v)v$ is the change in the surface energy of the body over unit time.

Now imagine an auxiliary motion which differs from the true motion in that the instantaneous velocity of the tip of the crack is zero and the wedge moves in such a way that at any instant of time the quantities l, $\dot{l}$ and $\ddot{l}$ are the same for both motions. The corresponding equation of conservation of energy is

$$\frac{d\mathscr{E}'}{dt} + \frac{d\Pi'}{dt} = -F\frac{dl}{dt} \tag{1.6}$$

where $\mathscr{E}'$, Π' and F' are, respectively, the kinetic and potential energies and the wedging force for the auxiliary motion. Since the velocity of the wedge is small as compared with the velocity of sound, we can accept the approximation that the quantities $d\Pi'/dt$ and F' are fully defined by the geometry of the motion, in this case by the quantities l, $\dot{l}$, ..., which are the same for both motions. Thus $F' = F$, $d\Pi'/dt = d\Pi/dt$, so that by subtracting (1.6) from (1.5) we obtain

$$\frac{d(\mathscr{E}-\mathscr{E}')}{dt} = [F - 2T(v)]\,v \qquad (1.7)$$

Let us now assess the order of the terms appearing in this equation. The elastic displacements of points in the body are evidently of the order of h and the strains of the order of h/l. Then the force F is of the order of the product of the normal stress on the sides of the wedge ($\sim Eh/l$) and the width of the wedge ($\sim h$), so that $F \sim Eh^2/l$. The quantity T is of the same order (cf. (1.4) and (1.2)). The particle velocities in the body are of the order of h/τ, where τ is the characteristic time of the process and $\tau \sim l/V$, so that the particle velocity is of the order of Vh/l. Thus the rate of change in kinetic energy is of the order of

$$\rho \frac{V^2h^2}{l^2} l^2 \frac{V}{l} \sim \rho \frac{V^3h^2}{l} \sim FV \frac{V^2}{c^2}$$

i.e. of the order of the terms on the right hand side of equation (1.7) multiplied by V^2/c^2.

This means that equation (1.7) contains a small parameter, so that with our assumption that $V/c \ll 1$, the term on the left-hand side of this equation is a small difference of large numbers. For the present approximation, therefore, we must consider only those terms with a small parameter which contain the highest derivatives (in this case the quantity $\ddot{l}$).

If, in particular, we assume that the dynamic part of the wedging force F is approximately independent of the acceleration of the tip of the crack, we find that the quantity F in equation (1.7) can be evaluated by means of a static solution. Bearing in mind that we have assumed the dimensions of the body to be very large (compared with the length of the free crack), the wedging force can be evaluated by means of a static solution for an infinite body, from the formula [3]

$$F = -2\int_{l}^{\infty} \sigma_y f_*'(x)\,dx$$

where σ_y is the normal stress exerted by the wedge on the surface of the crack and $f_*(x)$ is a function which defines the shape of the wedge. We shall calculate the value of the wedging force F for the more general case of a semi-infinite section along the positive semi-axis loaded on the surface by a symmetrical normal stress $\sigma_y = -g(x)$ distributed according to some law. We assume that the applied normal stress is such that the normal displacement of points on the surface of the section $f_*(x)$ increases monotonically with increase in x. We have

$$F = -2\int_0^\infty \sigma_y f_*'(x)\,dx \qquad \left(f_*'(x) = \frac{2(1-\nu^2)}{\pi E\sqrt{x}}\int_0^\infty \frac{g(t)\sqrt{t}\,dt}{t-x}\right)$$

(For the expression for $f_*'(x)$ see, for example, [7].) From this

$$F = \frac{4(1-\nu^2)}{\pi E}\int_0^\infty\int_0^\infty \frac{g(x)\,g(t)\,t\,dx\,dt}{\sqrt{x}\sqrt{t}\,(t-x)} =$$

$$= \frac{4(1-\nu^2)}{\pi E}\int_0^\infty \frac{g(x)}{\sqrt{x}}\left(\int_0^\infty \frac{g(t)\,(t-x+x)}{\sqrt{t}\,(t-x)}\,dt\right)dx = \frac{4(1-\nu^2)}{\pi E}\left(\int_0^\infty \frac{g(t)\,dt}{\sqrt{t}}\right)^2 - F$$

Thus we obtain

$$F = \frac{2\pi(1-\nu^2)N^2}{E}, \qquad N = -\frac{1}{\pi}\int_0^\infty \frac{g(t)\,dt}{\sqrt{t}}$$

where N is the coefficient of stress intensity [7]. It has been established that for equilibrium cracks $N = 0$, so that at the tip of the crack the stresses are finite and the closure of the edges of the crack is smooth. The condition for this is, of course, that the wedging force is zero. Note that the wedging force in this case is the "force needed to widen the crack" as proposed by Irwin [9].

The system of forces exerted by the wedge on the elastic body, and consequently, its resultant F, are independent of the forces acting in the end region and influencing the elastic field only in the immediate neighborhood of the tip of the crack. Therefore, in calculating F the quantity N should be taken equal to N_0 – the coefficient of stress intensity evaluated without taking into account cohesion forces. In the particular case of a sufficiently long wedge of constant thickness we can make use of the solution for a semi-infinite wedge in an infinite body, which gives:

$$F = \frac{Eh^2}{2\pi(1-\nu^2)\,l(t)}$$

3. In order to evaluate the left-hand side of equation (1.7), as a result of our assumption that the wedging velocity is small compared with the velocity of propagation of transverse oscillations, we can adopt the static solution. This gives an expression for the displacement in a moving system of coordinates:

$$\mathbf{u} = \{u_x, u_y\} = \mathbf{u}(\xi, \eta, l(t))$$

In the fixed system of coordinates xy attached to the body undergoing wedging, the velocity in the quasi-static approximation, by virtue of (1.1), is

$$\mathbf{q} = \frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial \xi} v(t) + \frac{\partial \mathbf{u}}{\partial l} \dot{l} \tag{1.8}$$

Also, the rate of change of kinetic energy is given by the expression

$$\frac{d\mathscr{E}}{dt} = \rho \int_{\Omega} \mathbf{q}\dot{\mathbf{q}}\, d\omega, \qquad \dot{\mathbf{q}} = \frac{d\mathbf{q}}{dt} \tag{1.9}$$

where the integration is taken through the whole volume Ω of the body. In evaluating the integral on the right-hand side of (1.9) we can, as before, in view of the assumption that $V/c \ll 1$, ignore terms which do not contain the highest derivative $\ddot{l}$. We then obtain

$$\frac{d\mathscr{E}}{dt} = Mv\ddot{l} + M_1\dot{l}\ddot{l} \tag{1.10}$$

$$M = \rho \int_{\Omega} \left[\left(\frac{\partial \mathbf{u}}{\partial \xi}\right)^2 + \left(\frac{\partial \mathbf{u}}{\partial \xi}\right)\left(\frac{\partial \mathbf{u}}{\partial l}\right)\right] d\omega, \qquad M_1 = \rho \int_{\Omega} \left[\left(\frac{\partial \mathbf{u}}{\partial \xi}\right)\left(\frac{\partial \mathbf{u}}{\partial l}\right) + \left(\frac{\partial \mathbf{u}}{\partial l}\right)^2\right] d\omega$$

Putting $v = 0$ in (1.10), we obtain an expression for $d\mathscr{E}'/dt$; subtracting this from (1.10), we find that

$$\frac{d(\mathscr{E} - \mathscr{E}')}{dt} = Mv\ddot{l} \tag{1.11}$$

Thus the determination of the left-hand side of equation (1.7) has been reduced to the evaluation of the "attached mass" M of the crack. Analysis shows that the attached mass, calculated from the static solution of the problem of the splitting of an infinite body by a semi-infinite wedge, is infinitely large. It is well-known that a similar difficulty exists in the determination of the attached masses of travelling dislocations (see, for example, the recent study by Weertman [10]). It is related to the specific nature of the plane stationary problem in the theory of elasticity and is obviated by the introduction of an external dimension of the body L. In the case under investigation, when the external dimension L of the body undergoing wedging is much greater than the length of the free crack l, there exists for the attached mass M an asymptotic expression in l/L:

$$M = \rho h^2 \left[A \ln \frac{L}{l} + B + o(1)\right] \tag{1.12}$$

The coefficient A can be calculated from the solution to the static problem of the splitting of an infinite body by a semi-infinite wedge. Assuming for simplicity that the wedge is of constant thickness, and employing the method of Muskhelishvili, we obtain the following expression for the displacement components u_x and u_y:

$$u_x + iu_y = \frac{h}{4\pi(1-\nu)}\{\varkappa\varphi(z) - \varphi(\bar{z}) - (z-\bar{z})\overline{\varphi'(z)}\} + u_0$$

$$z = \xi + i\eta, \qquad \varkappa = 3 - 4\nu$$

$$\varphi'(z) = \frac{1}{\sqrt{z(z-l)}}, \qquad \varphi(0) = 0, \qquad \lim_{z\to\infty}\frac{z}{\sqrt{z(z-l)}} = -1 \tag{1.13}$$

where the constant u_0 (which may depend on the length of the crack and, consequently, on time) is chosen so that the asymptotic solution (1.13) satisfies the condition that the velocity q vanishes at infinity. Taking into account (1.8), we obtain

$$u_0(t) = -\frac{(1-2\nu)}{2\pi(1-\nu)}\ln l(t) + \text{const} \tag{1.14}$$

where the constant is independent of time. The principal (logarithmic) term in the expression for M, i.e. the constant A, can be found by expanding the expression under the integral sign in the neighborhood of a point of infinity:

$$A = \frac{\varkappa^2 + 5}{16\pi(1-\nu)^2} \tag{1.15}$$

Assuming that the free crack is approximately equidistant from the boundaries of the body, we can also find the constant B from the asymptotic solution (1.13), carrying out the integration in (1.10) around circles with center at the tip of the crack.

This gives

$$B = A\ln\gamma, \qquad \gamma = \exp\left[\frac{\varkappa^2+1}{2(\varkappa^2+5)} + (2\ln 2)\frac{\varkappa^2-5}{\varkappa^2+5}\right] \tag{1.16}$$

Substituting this expression into (1.12) and neglecting small quantities $o(1)$, we obtain an expression for the attached mass of the crack in the form

$$M = m\rho h^2, \qquad m = \frac{\varkappa^2+5}{16\pi(1-\nu)^2}\ln\frac{\gamma L}{l}$$

Substituting the expressions found for the quantities F and $d(\mathscr{E} - \mathscr{E}')/dt$ into (1.7), expressing T in terms of the cohesion modulus according to formula (1.2) and cancelling the common factor v, we obtain the basic equation for determining $l(t)$ in the form:

$$mph^2\frac{d^2l}{dt^2}=F-2T=\frac{Eh^2}{2\pi(1-\nu^2)\,l\,(t)}-\frac{2\,(1-\nu^2)\,K^2\,(v)}{\pi E} \qquad (1.17)$$

4. If we introduce the force of inertia of the crack dI/dt applied at the tip in such a way that the work done by this force is equal to the decrement in $\mathscr{E}-\mathscr{E}'$, i.e.

$$\frac{d\,(\mathscr{E}-\mathscr{E}')}{dt}=-v\,\frac{dI}{dt} \qquad \text{or} \qquad -mph\frac{d^2l}{dt^2}=\frac{dI}{dt} \qquad (1.18)$$

then equation (1.17) can be looked upon as a force equality:

$$\frac{dI}{dt}=F-R, \qquad R=2T \qquad (1.19)$$

where $R=2T$ is the force resisting wedging, acting at the tip of the crack in the opposite direction to the travel of the crack.

Thus in a state of equilibrium or steady splitting the wedging force F is balanced by the resistance R, from which formula (1.4) can be derived. In the case of non-steady splitting the wedging force F is not balanced by the resistance R, since as a result of the inertia of the body the length $l(t)$ of the free crack has insufficient time to adapt itself to the cohesion modulus corresponding to the instantaneous velocity of the tip of the crack $v(t)$.

2. Investigation of the basic equation. 1. Taking $v=V+dl/dt$ as the independent variable, and l as the dependent, we reduce equation (1.17) to the form

$$\frac{dl}{dv}=\frac{(v-V)l}{A-BlK^2\,(v)}\equiv\frac{Pl}{Ql} \qquad \left(A=\frac{E}{2\pi\,(1-\nu^2)m\rho},\ B=\frac{2(1-\nu^2)}{\pi m\rho Eh^2}\right) \qquad (2.1)$$

Now during wedging the length l of the free crack varies only slightly (certainly insufficiently to affect the order) and therefore, since the assumption is that L/l is large and since it appears under the logarithm sign, the quantity m can be assumed to be constant.

We need consider the integral curves of equation (2.1) only in the first quadrant of the plane vl, since only the segments of the integral curves within this quadrant have any physical meaning. The length of the free crack in front of the wedge cannot be negative, and by virtue of the irreversibility of the crack the velocity of its tip v cannot be negative either.

Investigation shows that in the first quadrant equation (2.1) has only one singular point:

$$v = V, \qquad l = l_*(V) = \frac{E^2h^2}{4(1-\nu^2)^2 K^2(V)} \tag{2.2}$$

which corresponds to steady quasi-static wedging, when the wedge has a velocity V and the length of the crack in front of the wedge remains constant. An investigation of the nature of the singular point shows that it is a focal point or node, unstable when

$$K'(V) < 0 \tag{2.3}$$

and stable when $K'(V) > 0$.

Consequently, since the function $K(V)$ decays in the neighborhood of $V = 0$ (Fig. 2), there always exists a region of unstable steady wedging $0 < V < v_*$, where v_* is given by the equation

$$K'(v_*) = 0 \tag{2.4}$$

It is clear that if the curve $K(v)$ has no increasing portion within the range of velocities under consideration, then the region of unstable steady wedging extends at least over all velocities admitting a quasi-static treatment.

Consider the non-steady motion which occurs at a wedge velocity V which lies within the region of instability.

Figure 3 shows the isoclines $dl/dv = 0$ and $dl/dv = \infty$ for the case of instability of the singular point, together with the directions of motion of the mapping point along the integral curves.

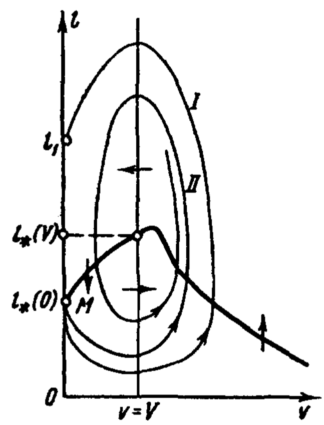

Fig. 3.

For a complete description of the motions which occur, it is necessary to establish what happens to the mapping point near the boundaries of the first quadrant. If the mapping point is located near the line $v = 0$, it reaches this line when $l > l_*(0)$ and leaves it in the first quadrant when $l \leqslant l_*(0)$. It follows from the irreversibility of the crack that in the former case the mapping point moves downwards along the line $v = 0$ with velocity V (the tip of the crack is stationary and the wedge is moving) until it reaches the point $M\{v = 0,\ l = l_*(0)\}$, after which it starts to travel along the integral curve starting from this point.

The boundary $l = 0$ is the only integral curve along which the mapping point can pass to infinity. Therefore the integral curves cannot leave the first quadrant through the boundary $l = 0$; on the other hand it is not difficult to see that motion along the integral curve $l = 0$ is unstable.

The mapping point leaving it within the first quadrant starts to move around the singular point, tending towards it or the limit cycle surrounding it, if such a cycle exists.

We can show that in the case of instability of the singular point, i.e. of "self-induced" oscillations, there is always at least one stable limit cycle.

The integral curve which leaves the point M can have one of two forms denoted in Fig. 3 by the numbers I and II. Note that this integral curve can intersect the isocline $dl/dv = \infty$ both to the right and to the left of the maximum. The latter is obviously the case for low wedging velocities and small slopes of the curve $K(v)$. If the integral curve leaving the point M is of the form I, then the limit cycle is a line consisting of a segment of this integral curve from the point M to the next point of intersection between it and the axis of ordinates, $v = 0$, $l = l_I$ with the closing segment the axis of ordinates $v = 0$, $l_*(0) \leqslant l \leqslant l_I$.

This limit cycle is always stable, independently of whether the singular point is stable or unstable.

This can readily be confirmed for points within the region which is internal with respect to the limit cycle by extending the segment of the boundary integral curve into this region from the point M. The integral curve extended in this way can either approach the singular point in a spiral (in the unstable case) or wind itself onto some inner limit cycle. An integral curve passing through any point in the internal region which is outside the second limit cycle, if such exists, must reach the axis of ordinates at some internal point of the closing segment $v = 0$, $l_*(0) \leqslant l \leqslant l_I$, and in so doing end up on the limit cycle. With respect to points in the outer region the stability of the limit cycle derives from the fact that the integral curve passing through every such point reaches the axis of ordinates above the point $l = l_I$ after which, as has already been pointed out, it falls with velocity V to the point $v = 0$, $l = l_I$ and ends up on the limit cycle. Note that in all cases the mapping point reaches the limit cycle over a finite interval of time.

If the integral curve starting from the point M is of the form II, then the existence of at least one absolutely stable limit cycle follows from the instability of the singular point [11].

Case I corresponds to oscillations with intervals when the tip of the crack is stationary (Fig. 4*a*) and the case II corresponds to oscillations without these stationary intervals (Fig. 4*b*).

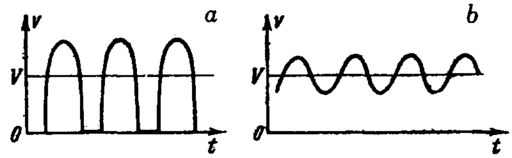

Fig. 4.

2. For the system (2.1) we have

$$-\left(\frac{\partial P}{\partial l}+\frac{\partial Q}{\partial v}\right) = BK^{2\prime}(v)$$

This expression does not vanish when $v < v_*$.

From the criterion of Bendixson [11] it follows that if the limit cycle is situated to the left of the line $v = v_*$, then it corresponds to oscillations with intervals when the tip of the crack is stationary.

Thus it has been shown that for $V > v_*$ (if the critical velocity v_* exists) steady wedging is stable with respect to small disturbances. For $V < v_*$ steady wedging is unstable and there exists a self-oscillatory regime of crack propagation. In the general case one is limited to conclusions of a qualitative nature; in order to calculate the self-oscillatory motions induced it is necessary to integrate equation (1.17) numerically with the function $K(v)$ specified in some definite way. The limit cycle is found by integration as a closed curve C in the plane of vl. The period of the oscillations is given by the formula

$$\theta = \oint_C \frac{dl}{v(l) - V} \tag{2.5}$$

The wavelength λ – the path travelled by the tip of the crack during one cycle – is given by the expression

$$\lambda = \int_0^\theta v(\tau)\, d\tau = V\theta \tag{2.6}$$

In practice the wavelength can be found, for example, as the distance between two neighboring ridges in the plane of the split.

It can be shown that the limit cycle corresponding to a certain value of the wedging velocity V includes all limit cycles corresponding to smaller values of V. Thus the amplitude of the oscillations of the length of the free crack increases with increase in the velocity of the wedge.

3. Limiting cases. Discussion of results. 1. Let us write equation (1.17) in non-dimensional form. We set

$$\Lambda = \frac{l}{l_*(0)}, \quad \tau = \frac{v_1 t}{l_*(0)}, \quad f\left[\frac{V}{v_1} + \frac{d\Lambda}{d\tau}\right] \equiv \frac{K^2(v)}{K^2(0)}, \quad \alpha = \pi(1-\nu)m\frac{v_1^2}{c^2}, \quad w = \frac{v}{v_I} \tag{3.1}$$

where v_1 is the characteristic velocity, which it is convenient to determine in different ways for different cases. The equation then becomes

$$\alpha\frac{d^2\Lambda}{d\tau^2} = \frac{1}{\Lambda} - f\left[\frac{V}{v_1} + \frac{d\Lambda}{d\tau}\right] \tag{3.2}$$

It has already been shown that the relation $K(v)$ is analogous to the relation between the coefficient of Coulomb friction and the relative velocity of oscillating surfaces. In the theory based on this assumption for self-oscillations in the presence of Coulomb friction developed by pupils of Mandel'shtam and Papaleksi, there are two important cases for which the investigation can be carried through to completion. In the first case the coefficient of friction depends very much on velocity, and if the inertia of the vibrating body is small enough, then the phase plane is divided into a region of "rapid" motions and a region of "slow" motions. If we replace the rapid motions by jumps, we need not in general take inertia into account. This case of discontinuous relaxational vibrations has been studied by Khaikin and Kaidanovskii [12]. In the second case the coefficient of friction depends very little on velocity and inertia must be taken into account. This case has been investigated by Strelkov [13]. Similar possibilities arise in the problem of self-oscillations during wedging. Consider the first of these - the case when the cohesion modulus K depends strongly on the velocity v. Putting $v_1 = V$ in (3.1) and transferring in equation (3.3) to the phase plane $w\Lambda$ we obtain

$$\frac{d\Lambda}{dw} = \frac{\alpha(w-1)\Lambda}{1-f(w)\Lambda}, \qquad \alpha = \pi(1-\nu)m\frac{V^2}{c^2} \tag{3.3}$$

The parameter α, which is a measure of the inertia of the process, was at the outset assumed to be small. We see from (3.3) that when α is small the quantity $d\Lambda/dw$ differs appreciably from zero only in a narrow strip surrounding the isocline $d\Lambda/dw = \infty$; elsewhere in the $w\Lambda$ plane $d\Lambda/dw$ is approximately zero for small values of α. Therefore, in accordance with the general results of the "boundary-layer" theory - the theory of equations with a small parameter in front of the highest derivative (see [11], Chapter 10) - we find that when the curve $f(w)$ has a maximum ($w = w_*$) the phase

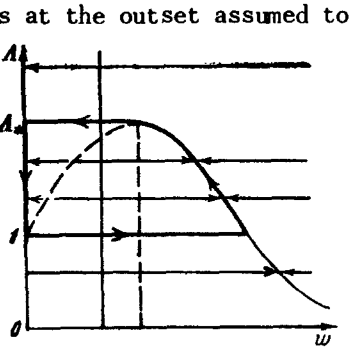

Fig. 5.

diagram of Fig. 3 in the limit as $\alpha \to 0$ assumes the form shown in Fig. 5.

On the basis of the general method outlined in Section 4 of Chapter 10 of [11] we obtain the following results. Slow motions occur in the region $1/\Lambda - f| < 0(w_0\alpha^{1/2})$, where $w_0 > 0$ is defined by the equation $f(w_0) = 1$. In the remaining part of the phase plane the motions are rapid. When $w > w_*$, the mapping point enters the region of slow motion, remains in this region, and in the limit moves in accordance with the equation

$$\frac{1}{\Lambda} = f\left(1 + \frac{d\Lambda}{d\tau}\right) \tag{3.4}$$

When $w < w_*$, the mapping point leaves the region of slow motions and enters the region of rapid variation in w at an almost constant value of Λ.

Consider the integral curve which starts from the point M. If

$$O\,(w_0 \sqrt{\alpha} \,/\, (1\,/\,f')_{w=0}') < 1$$

(the wedge velocity is not very high), then this curve will enter the region of rapid motions. Remaining approximately horizontal, it reaches the isocline $d\Lambda/dw = \infty$ at the point $(w_0,\ 1)$ and then travels along this isocline in the region of slow motions until it reaches the point $w = w_*$, $\Lambda = \Lambda_*$.

If

$$w_* - 1 > O\,(w_0^{1/3}\alpha^{1/4} \,/\, \sqrt{-(1\,/\,f)''_{w=w_*}})$$

(the wedge velocity is not too close to v_*) then the curve, after leaving the point M, enters the regions of rapid motions close to this point and, remaining almost horizontal, travels to the point $w = 0$, $\Lambda = \Lambda_*$, thus describing a curve of the type I.

Consequently, if the above inequalities are satisfied, which, taking into account (3.1), can be reduced to the condition

$$V_1 \ll V \ll V_2 \tag{3.5}$$

where V_1 and V_2 tend, respectively, to zero and v_* with increase in $K'(0)$ and $K''(v_*)$, then there exists a discontinuous limit cycle which is stable and unique. It consists of a curve of the type I as just described with a closing segment formed by the axis of ordinates, along which, as before, the non-dimensional velocity is equal to $d\Lambda/d\tau = 1$. The way in which the velocity v of the tip of the crack varies with time t is shown in Fig. 6.

In the zero-th approximation the period of the resulting discontinuous relaxational oscillations is the sum of the times taken by the mapping point to travel the non-horizontal portions of the limit cycle, and is determined by integrating equation (3.4). If the range of wedging velocities admitted by the inequalities (3.5) is so wide that V can assume values which are small as compared with v_*, then for such values of V the contribution to the period from the curvilinear portion of the limiting cycle can be ignored. The wavelength λ of the oscillations then ceases to depend on velocity, and becomes simply the distance between the lengths of the free crack corresponding to the minimum and initial cohesion moduli

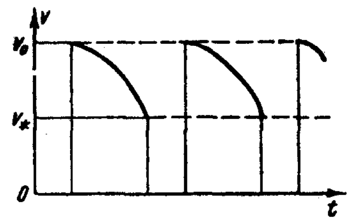

Fig. 6.

$$\lambda = \frac{[K^2(0) - K^2(v_*)]\, E^2 h^2}{4(1-\nu^2)^2 K^2(0)\, K^2(v_*)} \tag{3.6}$$

If over the whole interval $0 < V < v_*$ the inequalities (3.5) are not satisfied (this may be the case for sufficiently small values of $K'(0)$ and $K''(v_*)$), then the concept of relaxational oscillations becomes unsuitable and inertia must be taken into account. We shall not investigate this case fully here; we shall confine our attention to the case of sufficiently small values of V for which the left-hand part of the inequality (3.5) is not satisfied.

In this case it is not convenient to take $v_1 = V$, and we shall therefore consider v_1 to be some fixed velocity, for instance, a certain fraction of the velocity of sound c.

When $W = V/v_1 \to 0$ the singular point is either an unstable node (when $4\alpha < [f'(0)]^2$), or an unstable focus (when $4\alpha > [f'(0)]^2$).

If the singular point is a node, then there can be no inner limit cycles, however small the value of W, since otherwise they would intersect some trajectory starting from the node. Therefore the trajectory, starting from the point M, is of the form I and, together with the closing segment of the axis of ordinates, forms a unique limit cycle which is stable and corresponds to oscillations with pauses. The size of this limit cycle does not tend to zero together with W. This is due to the fact that in the case of a node, when $\alpha \to 0$ and $v_* \ll c$, the limit cycle tends to the discontinuous limit cycle considered in the last item, the shape of which is independent of W.

If the singular point is a focus, then for sufficiently small values of W the trajectory starting from the point M makes its first half-turns in the region where equation (3.2) is approximately linear, and with the closing segment of the axis of ordinates forms a limit cycle of the same type as in the case of a node. Other limit cycles (necessarily inner) can be situated only inside this limit cycle, which, however, is impossible since there equation (3.2) is almost linear. Obviously, in the case of a focus the dimensions of the limit cycle tend to zero together with W.

The fact that equation (3.2) is approximately linear in the region where the limit cycle is located enables us to find this cycle, and in particular, to determine the wavelength. After linearization equation (3.2) becomes

$$\alpha \frac{d^2\delta}{d\tau^2} + f'(0)\frac{d\delta}{d\tau} + \delta = -f'(0)W, \qquad \delta = \Lambda - 1 \tag{3.7}$$

Its solution, which corresponds to a trajectory starting from the point $M(\delta = 0,\ d\delta/d\tau = -W$ when $\tau = 0)$, in the case of a focus $(4\alpha > [f'(0)]^2)$ is of the form

$$\delta(\tau) = -W\left\{e^{n\tau}\left[2\alpha n\cos\omega\tau + \frac{1-2\alpha n^2}{\omega}\sin\omega\tau\right] - 2\alpha n\right\}$$

$$2\alpha n = -f'(0), \qquad 2\alpha\omega = \sqrt{4\alpha - [f'(0)]^2} \tag{3.8}$$

The time t_1 taken by the mapping point to travel along the curvilinear portion of the limit cycle can be found from the condition that it reaches the axis of ordinates, i.e. $d\delta/d\tau = -W$.

From this, taking into account (3.8) we obtain

$$\sqrt{\alpha}\,\omega e^{-n\tau_1} = \cos(\omega\tau_1 + \varepsilon), \qquad \tan\varepsilon = n/\omega \tag{3.9}$$

Equation (3.9) has an infinite number of solutions, from which, obviously, we must select the smallest positive solution.

Thus the wavelength of the oscillations is

$$\lambda = Vt_1 + l(t_1) - l_*(0) \tag{3.10}$$

where t_1 and $l(t_1)$ are given by formulas (3.8) and (3.9).

The results obtained enable us to explain the occurrence of self-oscillations during the wedging of brittle bodies, both crystalline and amorphous in which the relation $K(v)$ exists. At present very little is known about this relation. Even if it is established by experiment that the wedging of certain crystals is stable at high velocities, we cannot conclude that the crystals have a critical velocity v^* or that these high velocities exceed v_*. It was pointed out above that with increase in the wedge velocity the amplitude of the oscillations in the length of the free crack increases. Therefore, a smooth surface of the crack, which was taken to be a sign that the wedging was stable, could result not because the wedging velocity exceeded v_*, but because the amplitude of the oscillations in the length of the free crack exceeded the dimensions of the crystal, so that over the length of the crystal non-uniformity in the motion of the tip of the crack would not be noticeable. Kosevich found that as the wedge velocity was increased, the wavelength grew until finally the surface of the crack became smooth. For crystals a weak rather than a strong relation $K(v)$ is to be expected. On the other hand, for materials such as amorphous polymers this relation would appear to be strong, and if these materials have a pronounced viscosity it would seem that their critical velocity v_* is not high. It is with such materials that there are most likely to occur discontinuous relaxational oscillations with pauses in the motion of the tip of the crack, which are characteristic of a strong relation $K(v)$ which has a minimum.

Oscillations with pauses in the motion of the tip of the crack (relaxational and non-relaxational) evidently have a principal value. It could be that as a result of hardening the value of K depends on the duration of the pause. Here again there is an analogy with friction: the coefficient of friction can depend on the duration of contact of vibrating surfaces. The corresponding theory has been developed in a paper by Ishlinskii and Kragel'skii.

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